Skew left bracoids via abelian maps

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Outline

Recall that a *skew bracoid* is a quintuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot), (N, \star)$ are groups, and G acts transitively on N via $\odot$ such that

$$g \odot (\eta \star \mu) = (g \odot \eta) \star (g \odot 1_N)^{\star -1} \star (g \odot \mu)$$

for all $g \in G,\ \eta, \mu \in N$.

Frequent abbreviations:

- "bracoid" for "skew bracoid";
- $(G, N, \odot)$ (or (G, N)) for $(G, \cdot, N, \star, \odot)$;
- gh for $g \cdot h$;
- η^{-1} for $\eta^{\star -1}$;
- We refer to the expression above as the "bracoid relation".

Recall that a *skew bracoid* is a quintuple $(G, \cdot, N, \star, \odot)$, where $(G, \cdot), (N, \star)$ are groups, and G acts transitively on N via $\odot$ such that

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for all $g \in G,\ \eta, \mu \in N$.

Objective

Use the theory of abelian maps to construct examples of bracoids.

Abelian maps: a quick review

Let $G = (G, \cdot)$ be a (non abelian) group. An endomorphism $\psi : G \to G$ such that $\psi(G)$ is abelian is called an *abelian map*.

Any abelian map $\psi : G \to G$ creates a bi-skew brace structure $(G, \circ, \cdot)$ where

$$g \circ h = g\psi(g^{-1})h\psi(g).$$

It is well known that this construction can be generalized [Caranti-Stefanello, K–, Stefanello-Trappeniers] but won't be here.

Of course, an abelian map creates a bracoid structure with $G = (G, \cdot),\ N = (G, \circ),\ g \odot h = gh$, but can we use ψ to construct non-brace bracoids?

Outline

Let $\psi : G \to G$ be abelian, and define $\phi(g) = g\psi(g^{-1})$.

Note that ϕ is not an endomorphism, however it is a homomorphism $(G, \circ) \to (G, \cdot)$.

In particular, $\phi(G) \leq (G, \cdot)$: $\phi(g)\phi(h) = \phi(g\psi(g^{-1})h\psi(g))$.

We say a subgroup H of G is ψ-*compatible* (or *compatible* for short) if $[G, \phi(H)] \leq H$.

Suppose H is ψ-compatible. Let $N = G/H$ be the set of left cosets.

For $xH, yH \in N$ we define

$$xH \star yH = (x \circ y)H = x\psi(x^{-1})y\psi(x)H.$$

$[G,\phi(H)] \le H,\ xH \star yH = x\psi(x^{-1})y\psi(x)H$

Note that since

$$
\begin{aligned}
xh_1H \star yh_2H &= xh_1\psi(h_1^{-1}x^{-1})yh_2\psi(xh_1)H \\
&= x(h_1\psi(h_1^{-1}))\psi(x^{-1})yh_2\psi(x)(\psi(h_1)h_1^{-1})H \\
&= x\psi(x^{-1})yh_2\psi(x)[(\psi(x^{-1})yh_2\psi(x^{-1}))^{-1},\phi(h_1)]H \\
&= x\psi(x^{-1})yh_2\psi(x)H \\
&= x\psi(x^{-1})y\psi(x)\psi(x^{-1})h_2\psi(h_2^{-1})\psi(x)\psi(h_2)h_2^{-1}H \\
&= x\psi(x^{-1})y\psi(x)[\psi(x^{-1}),\phi(h_2)]H \\
&= x\psi(x^{-1})y\psi(x)H \\
&= xH \star yH
\end{aligned}
$$

the operation is well-defined.

$[G, \phi(H)] \leq H, \; xH \star yH = x\psi(x^{-1})y\psi(x)H$

Easy to show: $(G/H, \star)$ is a group, identity eH and $(xH)^{-1} = \psi(x)x^{-1}\psi(x^{-1})H$.

For $g \in G, xH \in G/H$ define $g \odot (xH) = gxH$.

This is clearly a transitive action.

Also, $(g \odot xH) \star (g \odot eH)^{-1} \star (g \odot yh) = gxH \star (gH)^{-1} \star gyH$ and

$$\begin{aligned} gxH \star gH^{-1} \star gyH &= gxH \star (\psi(g)g^{-1}\psi(g^{-1}))H \star gyH \\ &= gx\psi(gx)^{-1}(\psi(g)g^{-1}\psi(g^{-1}))\psi(g)gy\psi(g^{-1})\psi(gx)H \\ &= gx\psi(x)^{-1}y\psi(x)H \\ &= g \odot (xH \star yH) \end{aligned}$$

and the bracoid relation holds.

$g \cdot h = gh,\ xH \star yH = x\psi(x^{-1})y\psi(x)H,\ g \odot xH = gxH$

Theorem (K–, Truman, 2023)

With the notation as above, $(G, \cdot, G/H, \star, \odot)$ is a bracoid.

Remarks

- The requirement that H be ψ-compatible is both necessary and sufficient here.
- We do not require the group(s) to be finite.

Outline

A question

For a given $\psi : G \to G$, are there ψ-admissible subgroups?

(Recall we require $[G, \phi(H)] \leq H$.)

Non-interesting examples:

$H = \{1_G\}$ This gives the bi-skew brace $(G, \cdot, \circ)$ from the original abelian map construction. The bracoid form is $(G, \cdot, G, \circ, \cdot)$.

$H = G$ This gives the unique bracoid structure $(G, \{1_G\})$.

$[G, \phi(H)] \leq H$

Better question. For a given $\psi : G \to G$, are there proper, nontrivial ψ-admissible subgroups?

Some admissible subgroups are guaranteed.

$H = F$ where $F = \{f \in G : \psi(f) = f\}$, the set of fixed points.
For $f \in F$, $\phi(f) = f\psi(f^{-1}) = ff^{-1} = 1_G$ so
$[G, \phi(F)] = [G, \{1_G\}] = \{1_G\} \leq F$.
Unless ψ is fixed-point free, F will be proper and nontrivial. (Also admissible: any $F' \leq F$.)

$H = K$ where $K = \ker \psi$. For $k \in K$, $\phi(k) = k$ and since $K \trianglelefteq G$ we have $[G, \phi(K)] = [G, K] \leq K$.
The subgroup K is nontrivial, and $K = G$ if and only if ψ is the trivial map (which is not very interesting either).
(Also admissible: any $K' \leq K,\ K' \trianglelefteq G$.)

$H = KF$ where K, F as above. $[G, \phi(KF)] = [G, K] \leq K \leq KF$.
While almost always nontrivial, possible to have $KF = G$.

Outline

First example

Let $G = D_n = \langle r, s : r^n = s^2 = rsrs = 1_G \rangle$ with $n \geq 4$ even.

Pick $0 \leq j < n/2$ and define $\psi : G \to G$ by $\psi(r) = r^j s,\ \psi(s) = 1_G$.

Fact. The subgroups of D_n are all of the form

$$H_d := \langle r^d \rangle \text{ or } H_{c,d} := \langle r^d, r^c s \rangle;\ 0 \leq c < d,\ d \mid n.$$

Complete list of ψ-admissible subgroups:

- H_1
- H_{2i}
- $H_{c,1} = G$
- $H_{0,2} = K$
- $H_{c,d}$ with $d \mid 2(j - c)$, c odd and d even.

Note that $F = H_{j,n} = \langle r^c s \rangle$ if j is odd, $F = H_n = \{1_G\}$ if j is even.

Second example

Let $G = \langle a, b \rangle$ be the free group on two generators.

Define $\psi : G \to G$ by $\psi(a) = \psi(b) = b$.

Some admissible subgroups:

- $F = \langle b \rangle$
- $F_d = \langle b^d \rangle, \; d \geq 0$
- $K = \{\prod a^{r_i} b^{s_i} : \sum (r_i + s_i) = 0\}$
- $KF = G$
- $KF_d = \{\prod a^{r_i} b^{s_i} : \sum (r_i + s_i) \equiv 0 \pmod{d}\}$

This produces an infinite number of bracoids.

Outline

Note: K is no longer $\ker\psi$.

Let E/K be Galois, $\mathrm{Gal}(E/K) = (G, \cdot)$.

Let $\psi : G \to G$ be abelian, and let $H \leq G$ be ψ-compatible.

Let $L = E^H$.

Then the bracoid $(G, \cdot, G/H, \star, \odot)$ gives a Hopf-Galois structure on L/K.

Explicitly, let $N = \{\lambda_\star(gH) : gH \in (G/H, \star)\}$, i.e., $\lambda_\star(gH)[xH] = g\psi(g^{-1})x\psi(g)H$.

Then N is a regular, G-stable subgroup (note that ${}^k\lambda_\star(gH) = \lambda_\star(kg\psi(g^{-1})k\psi(g)))$, and hence gives a Hopf-Galois structure on L/K.

$\lambda_\star(gH)[xH] = g\psi(g^{-1})x\psi(g)H$

Case $H = \ker\psi$. We have $(G/H, \star)$ is simply the usual factor group G/H and $\lambda_\star(gH)$ is left multiplication by gH. This is the canonical nonclassical Hopf-Galois structure on the Galois extension L/K.

Case $H = F$. Here, L/K is not Galois unless $\phi(G) \leq \mathrm{Cent}(F)$.

Also, if $(G/H, \star)$ is nonabelian, then we can construct another regular, G-stable subgroup $P = \{\rho_\star(gH) : gH \in G/H\}$, i.e., $\rho_\star(gH)[xH] = x\psi(x^{-1})g\psi(x)H$, so there is a “compatibility with opposites”.

Let E/K be an $G := S_4$-extension.
Define $\psi : G \to G$ by

$$\psi(\sigma) = \begin{cases} 1_G & \sigma \in A_4 \\ (12) & \sigma \notin A_4 \end{cases}.$$

Then ψ is abelian. The list of ψ-compatible subgroups:

ψ-subgroup H	$(G/H, \star)$	$[E^H : K]$	
$\{1_G\}$	$A_4 \times C_2$	24	HGS from ψ on S_4
$\langle (12) \rangle$	A_4	12	$H = F$
$\langle (12)(34), (13)(24) \rangle$	C_6	6	$H = K$
$\langle V, (34) \rangle$	C_3	3	$\lambda(G) \leq \mathrm{Perm}(G/H)$
$\langle V, (234) \rangle$	C_2	2	$\lambda(G) \leq \mathrm{Perm}(G/H)$
S_4	triv.	1	Not interesting

Note: blue are the cases where E^H/K is Galois.

Outline

What's next?

Things to think about going forward.

1. How common are ψ-compatible subgroups?
   Early evidence: not very, at least for G finite (and nonabelian).
   For $\psi : S_n \to S_n$ analogous to the map above, the ψ-compatible subgroups in $S_n,\ n = 5, 6$ are $\{1_G\}, \langle(12)\rangle, A_n$ and S_n.
2. Can this be generalized to larger categories of endomorphisms, e.g., $\psi : G \to G$ with $\psi([G, G]) \leq Z(G)$?
3. Are there examples of $\psi : G \to G$ with a ψ-compatible $H \trianglelefteq G$ where the resulting HGS couldn't arise from some abelian map on G/H?
4. Is there value to infinite bracoids?

Thank you.